

ASYMPTOTICS FOR SOLUTIONS OF ELLIPTIC EQUATIONS IN DOUBLE DIVERGENCE FORM

VLADIMIR MAZ'YA AND ROBERT MCOWEN

ABSTRACT. We consider weak solutions of an elliptic equation of the form $\partial_i \partial_i (a_{ij} u) = 0$ and their asymptotic properties at an interior point. We assume that the coefficients are bounded, measurable, complex-valued functions that stabilize as $x \rightarrow 0$ in that the norm of the matrix $(a_{ij}(x) - \delta_{ij})$ on the annulus $B_{2r} \setminus B_r$ is bounded by a function $\Omega(r)$, where $\Omega^2(r)$ satisfies the Dini condition at $r = 0$, as well as some technical monotonicity conditions; under these assumptions, solutions need not be continuous. Our main result is an explicit formula for the leading asymptotic term for solutions with at most a mild singularity at $x = 0$. As a consequence, we obtain upper and lower estimates for the L^p -norm of solutions, as well as necessary and sufficient conditions for solutions to be bounded or tend to zero in L^p -mean as $r \rightarrow 0$.

0. INTRODUCTION

We are interested in the local behavior of weak solutions to the elliptic equation in “double divergence form”

$$(1) \quad \mathcal{A}u := \partial_i \partial_j (a_{ij}(x)u(x)) = 0,$$

where we have used $\partial_i = \partial/\partial x_i$ and the summation convention; the coefficients $a_{ij} = a_{ji}$ are bounded, measurable, complex-valued functions in a domain to be specified. The operator \mathcal{A} arises naturally as the formal adjoint \mathcal{L}^* of the operator in “non-divergent form,”

$$(2) \quad \mathcal{L} = \bar{a}_{ij}(x) \partial_i \partial_j.$$

Solutions of (1) are not only important for the solvability of $\mathcal{L}u = f$, but for properties of the Green's function for \mathcal{L} . When the coefficients a_{ij} are real-valued functions, the operators \mathcal{L} and \mathcal{A} have been studied by Sjögren ([24]), Bauman ([3], [4], [5]), Fabes and Stroock ([12]), Fabes, Garofalo, Marín-Malavé, and Salsa ([11]), Escauriaza and Kenig ([10]), and Escauriaza ([8], [9]); these papers use techniques, such as the maximum principle, that rely on the coefficients being real-valued. Not only do our techniques apply to complex-valued coefficients, but they give additional information for real-valued coefficients; we shall explain this in some detail at the end of this Introduction.

We want to study weak solutions of (1) in a neighborhood of an interior point of the domain, say $x = 0$, where the coefficients a_{ij} approach δ_{ij} in the sense that

$$(3) \quad \sup_{r < |x| < 2r} |\mathbf{a}(x) - \mathbf{I}| \leq \Omega(r);$$

here $\mathbf{a}(x)$ is the matrix (a_{ij}) , \mathbf{I} is the identity matrix, $|\cdot|$ denotes the matrix norm, and $\Omega(r) \rightarrow 0$ as $r \rightarrow 0$ in a manner that we shall describe. We remark that, when the coefficients are real-valued, the more general case obtained by replacing δ_{ij} by constants α_{ij} that satisfy the ellipticity condition can be reduced to (3) by means of an affine change of the x variables. Of course, this

reduction of the more general case to (3) is not available when the constants α_{ij} are complex-valued, but we have chosen to treat the special case $\alpha_{ij} = \delta_{ij}$ in order to take advantage of technical simplifications in the formulations and proofs of our results.

The specific hypotheses that we impose on the function $\Omega(r)$ in (3) are as follows:

$$(4) \quad \int_0^1 \frac{\Omega^2(t)}{t} dt < \infty,$$

$$(5) \quad \Omega(r) r^{-1+\varepsilon} \quad \text{is nonincreasing for } 0 < r < 1 \text{ and,}$$

$$(6) \quad \Omega(r) r^{n-\varepsilon} \quad \text{is nondecreasing for } 0 < r < 1;$$

here $\varepsilon > 0$. Clearly, (4) together with (5) or (6) implies that $\Omega(r) \rightarrow 0$ as $r \rightarrow 0$, so the coefficients a_{ij} are approaching δ_{ij} as $x \rightarrow 0$, although perhaps at a slow rate.

A weak solution of (1) in a domain $U \subset \mathbb{R}^n$ is a function $u \in L^1_{loc}(U)$ that satisfies

$$(7) \quad \int_U a_{ij}(x) u(x) \partial_j \partial_i \eta(x) dx = 0 \quad \text{for all } \eta \in C_0^\infty(U).$$

Weak solutions of (1) need not be continuous under our assumptions on the coefficients, so to measure growth or decay as $x \rightarrow 0$, we will use the mean in L^p for some $p \in (1, \infty)$:

$$(8) \quad M_p(w, r) := \left(\int_{A_r} |w|^p dx \right)^{1/p},$$

where A_r is the annulus $B_{2r} \setminus B_r$ with $B_r = \{x \in \mathbb{R}^n : |x| < r\}$; here (and elsewhere in this paper) the slashed integral denotes the mean value. (We will also use the notation $M_p(w, r)$ when w is vector or matrix valued; in this case, $|w|$ denotes the norm of w .)

For our local results, we will consider (1) in the unit ball B_1 and we will assume

$$(9) \quad \int_0^1 \frac{\Omega^2(t)}{t} dt < \delta,$$

where δ is sufficiently small. In fact, this represents no additional assumption on $\Omega(r)$ since we could replace B_1 in what follows by a very small ball B_γ in order to make the integral $\int_0^\gamma \Omega^2(t) t^{-1} dt$ as small as necessary.

At times it will be useful to consider solutions of (1) in all of \mathbb{R}^n ; in that case, we assume that $a_{ij} = \delta_{ij}$ outside of B_1 . Our first result concerns such a solution.

Theorem 1. *Let $n \geq 2$, $p \in (1, \infty)$, and $\Omega(r)$ satisfy (5), (6), and (9). There exists a weak solution $Z \in L^p_{loc}(\mathbb{R}^n)$ of equation (1) in \mathbb{R}^n satisfying*

$$(10) \quad Z(x) = \exp \left[-\frac{1}{|\partial B_1|} \int_{B_1 \setminus B_{|x|}} (a_{ii}(y) - n a_{ij}(y) y_i y_j |y|^{-2}) \frac{dy}{|y|^n} \right] (1 + \zeta(x)),$$

where ζ satisfies

$$M_p(\zeta, r) \leq c \max \left(\Omega(r), \int_0^r \frac{\Omega^2(t)}{t} dt \right) \quad \text{for } 0 < r < 1.$$

Remark 1. *The proof of Theorem 1 shows that Z has a limit at infinity, $Z(\infty)$, satisfying*

$$|Z(x) - Z(\infty)| \leq c \sqrt{\delta} |x|^{-n}.$$

But we are more interested in the behavior of Z at the origin; Corollary 3 below can be used to show that Z has at most a mild singularity at the origin (cf. (19)).

Our second theorem uses the solution Z from Theorem 1 to characterize the asymptotics (as $x \rightarrow 0$) of weak solutions of (1); because this is a local result, we consider a solution in B_1 .

Theorem 2. *Let $n > 2$, $p \in (1, \infty)$, and $\Omega(r)$ satisfy (5), (6), and (9). Suppose that $u \in L^p_{\text{loc}}(\overline{B}_1 \setminus \{0\})$ is a weak solution of (1) in B_1 subject to the growth condition*

$$(11) \quad M_p(u, r) \leq c r^{2-n+\varepsilon_0},$$

where $\varepsilon_0 > 0$. Then there exists a constant C (depending on u) such that

$$(12) \quad u(x) = CZ(x) + w(x),$$

where the remainder term w satisfies

$$(13) \quad M_p(w, r) \leq c r^{1-\varepsilon_1}$$

for $0 < r < 1$ and any $\varepsilon_1 > 0$.

Remark 2. *The restriction $n > 2$ in Theorem 2 is caused by the existence of solutions for the Laplacian with logarithmic growth at $x = 0$ when $n = 2$. A refinement of the techniques used in proving Theorem 2 would be required to cover the case $n = 2$.*

The following two results are immediate consequences of Theorems 1 and 2.

Corollary 1. *Under the hypotheses of Theorem 2, the condition*

$$(14) \quad \liminf_{r \rightarrow 0} \int_{B_1 \setminus B_r} \text{Re}(a_{ii}(y) - na_{ij}(y)y_i y_j |y|^{-2}) \frac{dy}{|y|^n} > -\infty$$

is necessary and sufficient for the boundedness of $M_p(u, r)$ as $r \rightarrow 0$ for all $u \in L^p_{\text{loc}}(\overline{B}_1 \setminus \{0\})$ which are weak solutions of (1) and satisfy (11).

Corollary 2. *Under the hypotheses of Theorem 2, the condition*

$$(15) \quad \lim_{r \rightarrow 0} \int_{B_1 \setminus B_r} \text{Re}(a_{ii}(y) - na_{ij}(y)y_i y_j |y|^{-2}) \frac{dy}{|y|^n} = +\infty$$

is necessary and sufficient for $M_p(u, r) \rightarrow 0$ as $r \rightarrow 0$ for all $u \in L^p_{\text{loc}}(\overline{B}_1 \setminus \{0\})$ which are weak solutions of (1) and satisfy (11).

Moreover, we can use Theorems 1 and 2 to derive upper and lower estimates for the L^p -mean of solutions. Since $(\text{Re}(a_{ij}) - \delta_{ij})$ is a symmetric real matrix, it is not difficult to verify that

$$(16) \quad -2(n-1)|\text{Re} \mathbf{a}(y) - \mathbf{I}| \leq \text{Re}(a_{ii}(y) - na_{ij}(y)y_i y_j |y|^{-2}) \leq 2(n-1)|\text{Re} \mathbf{a}(y) - \mathbf{I}|.$$

Using $|\text{Re} \mathbf{a}(y) - \mathbf{I}| \leq |\mathbf{a}(y) - \mathbf{I}|$ and (3), we find that (10) also yields the following upper and lower estimates.

Corollary 3. *The solution Z in Theorem 1 satisfies for $r \in (0, 1)$*

$$(17) \quad c_1 \exp \left(-2(n-1) \int_r^1 \Omega(s) \frac{ds}{s} \right) \leq M_p(Z, r) \leq c_2 \exp \left(2(n-1) \int_r^1 \Omega(s) \frac{ds}{s} \right),$$

where c_1, c_2 are positive constants.

Even when the coefficients a_{ij} are real, the upper and lower bounds (17) appear to be new.

The principal analytic content of our results is contained in Theorem 1. The method of its proof is independent of, but related to, the asymptotic theory developed in [17]. In particular, L_p -means of type (8) were extensively used in [15] and [16]. The asymptotic formula that we obtain is analogous to that of [18], where an asymptotic representation near the boundary was obtained for solutions to the Dirichlet problem for elliptic equations in divergence form with discontinuous coefficients.

Now let us turn to the comparison of our results with the extensive work of the authors cited in the first paragraph; we refer to the excellent exposition in Escauriaza [9] for a more detailed description of these results and references to the literature. When \mathcal{L} is uniformly elliptic with real-valued, measurable, and bounded (although not necessarily continuous) coefficients on \mathbb{R}^n , the previous work shows: i) any nonnegative weak solution $Z \in L^1_{loc}(B_1)$ of (1) in B_1 satisfies $Z \in L^{n/(n-1)}(B_1)$ (cf. [12]); ii) for a fixed nontrivial nonnegative weak solution $Z \in L^1_{loc}(B_1)$ of (1) in B_1 , and every weak solution $u \in L^1_{loc}(B_1)$ of (1) in B_1 , the function u/Z is Hölder continuous in B_1 (cf. [3]); and iii) the existence of a unique nonnegative weak solution Z of (1) in \mathbb{R}^n that satisfies $\int_{B_1} Z dx = |B_1|$ and may be used to estimate the Green's function for the operator \mathcal{L} on \mathbb{R}^n (cf. [9]). These previous results are quite general, but do not apply to the case of complex coefficients that we consider because they rely upon the maximum principle. Moreover, even when the coefficients are real-valued, our results are somewhat different in nature than the previous ones: instead of estimates, we have obtained an asymptotic description of Z near a point where the a_{ij} are continuous in the sense of (3). For a more direct comparison, when the a_{ij} are real-valued and continuous at 0, Escauriaza [9] has obtained upper and lower estimates for the L^1 -norm of Z : for any $\gamma > 0$ there exists a constant N_γ such that

$$(18) \quad N_\gamma^{-1} r^\gamma \leq \int_{B_r} Z dx \leq N_\gamma r^{-\gamma} \quad \text{for } 0 < r < 1.$$

When the a_{ij} satisfy our stronger sense of continuity (3), the explicit upper and lower bounds (17) imply L^p -bounds analogous to (18). In fact, as we shall see in Section 2, the assumptions on $\Omega(r)$ imply that $\Omega(r) \leq c\sqrt{\delta}$ for $0 < r < 1$ (cf. (46)), so we easily obtain from (17)

$$(19) \quad N_\gamma^{-1} r^\gamma \leq M_p(Z, r) \leq N_\gamma r^{-\gamma} \quad \text{for } 0 < r < 1.$$

More importantly, however, our necessary and sufficient conditions for solutions to be bounded in L^p -mean as $r \rightarrow 0$ (Corollary 1) or tend to zero in L^p -mean as $r \rightarrow 0$ (Corollary 2) have not been obtained previously.

1. PRELIMINARY ESTIMATES

In this and the next section, we will use the spherical mean of a function w . For notational convenience, we denote the spherical mean using an “overbar”:

$$(20) \quad \overline{w}(r) = \int_{\partial B_1} w(r\theta) ds.$$

This should cause no confusion with complex conjugation since we will not have occasion to use the latter in these two sections. In particular, in this section we are concerned with solving an equation of the form

$$(21) \quad -\Delta v = \partial_i \partial_j (F_{ij}) - \overline{\partial_i \partial_j (F_{ij})} \quad \text{in } \mathbb{R}^n.$$

Here $F_{ij} \in L^1_{loc}(\mathbb{R}^n)$ and derivatives are interpreted in the sense of distributions. The norm of the matrix $\mathcal{F} = (F_{ij})$ will be denoted by $|\mathcal{F}|$.

Proposition 1. *Suppose that $F_{ij} \in L^p_{loc}(\mathbb{R}^n \setminus \{0\})$ satisfies*

$$(22) \quad \int_{|x|<1} |\mathcal{F}(x)| dx + \int_{|x|>1} |\mathcal{F}(x)| |x|^{-n-1} dx < \infty.$$

Then there exists a weak solution $v \in L^p_{loc}(\mathbb{R}^n \setminus \{0\})$ of (21) that satisfies

$$(23) \quad M_p(v, r) \leq c \left(\tilde{M}_p(\mathcal{F}, r) + r \int_{|x|>r} |\mathcal{F}(x)| |x|^{-n-1} dx + r^{-n} \int_{|x|<r} |\mathcal{F}(x)| dx \right),$$

where c is independent of r and we have introduced

$$\tilde{M}_p(w, r) := \left(\int_{r/2 < |x| < 4r} |w(x)|^p dx \right)^{1/p}.$$

Proof: It suffices to prove the result for $F_{ij} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ since the general case can be handled by a standard approximation argument. The function v is defined by convolution with Γ , the fundamental solution for the Laplacian:

$$v = \Gamma \star (\partial_i \partial_j F_{ij} - \overline{\partial_i \partial_j F_{ij}}).$$

Using $\int_{\mathbb{R}^n} f(y) \overline{g}(|y|) dy = \int_{\mathbb{R}^n} \overline{f}(|y|) g(y) dy$, we can write this as

$$v(x) = \int_{\mathbb{R}^n} \left(\Gamma(|x - y|) - \overline{\Gamma(|x - \cdot|)}(|y|) \right) \partial_i \partial_j F_{ij}(y) dy.$$

Now to compute the spherical mean of the fundamental solution, we can use the mean value theorem for harmonic functions to conclude that

$$\overline{\Gamma(|x - \cdot|)}(|y|) = \Gamma(\max\{|x|, |y|\}).$$

This enables us to express v as

$$v(x) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^n} \Gamma(|x - y|) F_{ij}(y) dy - \Gamma(|x|) \int_{|y| < |x|} \frac{\partial^2 F_{ij}}{\partial y_i \partial y_j} dy - \int_{|y| > |x|} \Gamma(|y|) \frac{\partial^2 F_{ij}}{\partial y_i \partial y_j} dy.$$

Now, integration by parts yields

$$\int_{|y| < |x|} \frac{\partial^2 F_{ij}}{\partial y_i \partial y_j} dy = \int_{|y|=|x|} \frac{\partial F_{ij}}{\partial y_j} \frac{y_i}{|y|} dS_y,$$

and

$$\begin{aligned} \int_{|y| > |x|} \Gamma(|y|) \frac{\partial^2 F_{ij}}{\partial y_i \partial y_j} dy &= - \int_{|y| > |x|} \Gamma'(|y|) \frac{y_i}{|y|} \frac{\partial F_{ij}}{\partial y_j} dy - \int_{|y|=|x|} \Gamma(|y|) \frac{\partial F_{ij}}{\partial y_j} \frac{y_i}{|y|} dS_y \\ &= \int_{|y| > |x|} \frac{\partial}{\partial y_j} \left(\Gamma'(|y|) \frac{y_i}{|y|} \right) F_{ij}(y) dy + \int_{|y|=|x|} \left(\Gamma'(|y|) \frac{y_j}{|y|} F_{ij}(y) - \Gamma(|y|) \frac{\partial F_{ij}}{\partial y_j} \right) \frac{y_i}{|y|} dS_y. \end{aligned}$$

Consequently,

$$\begin{aligned} v(x) &= \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^n} \Gamma(|x - y|) F_{ij}(y) dy - \int_{|y| > |x|} \left(\frac{\partial^2}{\partial y_j \partial y_i} \Gamma(|y|) \right) F_{ij}(y) dy \\ &\quad - \Gamma'(|x|) \int_{|y|=|x|} \frac{y_i y_j}{|y|^2} F_{ij}(y) dS_y. \end{aligned}$$

Now introduce χ_0 and χ_∞ as the characteristic functions of $B_{r/2}$ and B_{4r}^c , and let $\chi_1 = 1 - \chi_0 - \chi_\infty$ be the characteristic function of the annulus $\tilde{A}_r := B_{4r} \setminus B_{r/2}$. Then

$$\begin{aligned} v(x) - \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^n} \Gamma(|x - y|) (\chi_1 F_{ij})(y) dy &= \\ \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i \partial x_j} \Gamma(|x - y|) (\chi_0 F_{ij})(y) dy + \int_{\mathbb{R}^n} \frac{\partial^2}{\partial y_i \partial y_j} (\Gamma(|x - y|) - \Gamma(|y|)) (\chi_\infty F_{ij})(y) dy \\ &\quad - \int_{B_{4r} \setminus B_{|x|}} \left(\frac{\partial^2}{\partial y_i \partial y_j} \Gamma(|y|) \right) F_{ij}(y) dy - \Gamma'(|x|) \int_{|y|=|x|} \frac{y_i y_j}{|y|^2} F_{ij}(y) dS_y. \end{aligned}$$

We can estimate the four integral kernels and obtain that the right hand side is bounded by

$$c \left(\frac{1}{|x|^n} \int_{B_{r/2}} |F_{ij}(y)| dy + |x| \int_{B_{4r}^c} \frac{|F_{ij}(y)| dy}{|y|^{n+1}} + |x| \int_{B_{4r} \setminus B_{|x|}} \frac{|F_{ij}(y)| dy}{|y|^{n+1}} + \overline{|F_{ij}|}(|x|) \right)$$

This provides us with the following pointwise bound:

$$(24) \quad \left| v(x) - \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^n} \Gamma(|x-y|) (\chi_1 F_{ij})(y) dy \right| \leq c \left(\overline{|F_{ij}|}(|x|) + |x| \int_{B_r^c} |F_{ij}(y)| \frac{dy}{|y|^{n+1}} + |x|^{-n} \int_{B_r} |F_{ij}(y)| dy \right).$$

Using the L^p -boundedness of singular integral operators on \mathbb{R}^n (see [27]), we have

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^n} \Gamma(|x-y|) (\chi_1 F_{ij})(y) dy \right\|_{L^p(A_r)} \leq \|\chi_1 \mathcal{F}\|_{L^p(\mathbf{R}^n)} = \|\mathcal{F}\|_{L^p(\tilde{A}_r)}.$$

Elementary estimates may be applied to the remaining terms in (24) to obtain (23), completing the proof. \square

The integrals in (23) can be estimated in terms of M_p . For example, we substitute

$$|x|^{-n-1} = c_n \int_{\frac{|x|}{2} < |y| < |x|} |y|^{-2n-1} dy,$$

into the first integral and change order of integration to obtain the estimate

$$r \int_{|x|>r} |\mathcal{F}(x)| |x|^{-n-1} dx \leq c r \int_{|y|>r/2} \int_{|y|<|x|<2|y|} |\mathcal{F}(x)| dx \frac{dy}{|y|^{2n+1}} \leq c r \int_{r/2}^{\infty} M_p(\mathcal{F}, \rho) \frac{d\rho}{\rho^2}.$$

Similarly, we can show

$$r^{-n} \int_{|x|<r} |\mathcal{F}(x)| dx \leq c r^{-n} \int_0^r M_p(\mathcal{F}, \rho) \rho^{n-1} d\rho$$

and

$$\tilde{M}_p(\mathcal{F}, r)^p \leq c r^{-n} \int_{r/2}^{4r} M_p(\mathcal{F}, \rho)^p \rho^{n-1} d\rho.$$

Elementary estimates show that terms involving integration over $r/2 < \rho < r$ and $2r < \rho < 4r$ can be respectively dominated by the terms involving integration over $0 < \rho < r$ and $\rho > r$, so we obtain the following.

Corollary 4. *Under the hypotheses of Proposition 1, the weak solution v obtained there satisfies*

$$(25) \quad M_p(v, r) \leq c \left(r \int_r^{\infty} M_p(\mathcal{F}, \rho) \rho^{-2} d\rho + r^{-n} \int_0^r M_p(\mathcal{F}, \rho) \rho^{n-1} d\rho \right).$$

2. PROOF OF THEOREM 1

We shall prove Theorem 1 by reducing the problem of finding Z to solving an operator equation of the form $(I + \tilde{T})V = f$, where V and f are elements of a Banach space X of functions on $\mathbb{R}^n \setminus \{0\}$, and \tilde{T} is an integral operator of small norm on X . However, this reduction will take a few steps. To begin, let $r = |x|$, $\theta = x/|x|$, and $\eta \in C_0^\infty((0, \infty))$ be arbitrary. For $Z \in L_{loc}^p(\mathbb{R}^n)$ to be a weak solution of (1), we must have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \partial_i \partial_j \eta(|x|) a_{ij}(x) Z(x) dx \\ &= \int_0^\infty \left(\eta''(r) \int_{\partial B_1} Z(r\theta) a_{ij}(r\theta) \theta_i \theta_j ds + \frac{\eta'(r)}{r} \int_{\partial B_1} Z(r\theta) (a_{ii}(r\theta) - a_{ij}(r\theta) \theta_i \theta_j) ds \right) r^{n-1} dr, \end{aligned}$$

where ds denotes surface measure on the unit sphere, ∂B_1 . Hence,

$$\begin{aligned} 0 &= \int_0^\infty \eta'(r) \left(-\frac{d}{dr} \left[r^{n-1} \int_{\partial B_1} Z(r\theta) a_{ij}(r\theta) \theta_i \theta_j ds \right] \right. \\ &\quad \left. + r^{n-2} \int_{\partial B_1} Z(r\theta) (a_{ii}(r\theta) - a_{ij}(r\theta) \theta_i \theta_j) ds \right) dr, \end{aligned}$$

where the derivative is understood in the distributional sense. This implies

$$(26) \quad -r^{n-1} \frac{d}{dr} \int_{\partial B_1} Z(r\theta) a_{ij}(r\theta) \theta_i \theta_j ds + r^{n-2} \int_{\partial B_1} Z(r\theta) (a_{ii}(r\theta) - n a_{ij}(r\theta) \theta_i \theta_j) ds = C,$$

where C is an arbitrary constant. In what follows, we will take $C = 0$; as we shall see, the solution that we construct will in fact be a weak solution of (1) on all of \mathbb{R}^n , not just $\mathbb{R}^n \setminus \{0\}$. (See also the Remark at the end of this section.)

Let us introduce

$$(27) \quad v(r\theta) := Z(r\theta) - \overline{Z}(r),$$

where \overline{Z} is the spherical mean as in (20). We may now express (26) as

$$(28) \quad y'(r) + \frac{Q(r)}{r} y(r) = \frac{1}{r} K v(r),$$

where

$$(29) \quad y(r) := \oint_{\partial B_1} Z(r\theta) a_{ij}(r\theta) \theta_i \theta_j ds,$$

and

$$(30) \quad Q(r) := n - \frac{\alpha_0(r)}{\alpha(r)},$$

with

$$(31) \quad \alpha_0(r) := \oint_{\partial B_1} a_{ii}(r\theta) ds \quad \text{and} \quad \alpha(r) := \oint_{\partial B_1} a_{ij}(r\theta) \theta_i \theta_j ds;$$

in (28) we also have used

$$(32) \quad K v(r) := \oint_{\partial B_1} v(r\theta) a_{ii}(r\theta) ds - \frac{\alpha_0(r)}{\alpha(r)} \oint_{\partial B_1} v(r\theta) a_{ij}(r\theta) \theta_i \theta_j ds.$$

It follows from (3) that $|\alpha_0(r) - n| \leq c\Omega(r)$, $|\alpha(r) - 1| \leq c\Omega(r)$, and

$$(33) \quad |Q(r)| \leq c\Omega(r),$$

so $Q(r) \rightarrow 0$ as $r \rightarrow 0$. Since $\bar{v}(r) = 0$, we can also write (30) as

$$Kv(r) = \oint_{\partial B_1} v(r\theta)(a_{ii}(r\theta) - n) ds - \frac{\alpha_0(r)}{\alpha(r)} \oint_{\partial B_1} v(r\theta)(a_{ij}(r\theta) - \delta_{ij})\theta_i\theta_j ds.$$

In this last form it is evident that K satisfies

$$(34) \quad M_p(Kv, r) \leq c\Omega(r) M_p(v, r).$$

To obtain another equation involving y and v , we start from the identity

$$(35) \quad \Delta v = \overline{\partial_i \partial_j ((a_{ij} - \delta_{ij})v)} - \partial_i \partial_j ((a_{ij} - \delta_{ij})v) + \overline{\partial_i \partial_j ((a_{ij} - \delta_{ij})\bar{Z})} - \partial_i \partial_j ((a_{ij} - \delta_{ij})\bar{Z}).$$

Noting that

$$y(r) = \alpha(r)\bar{Z}(r) + \oint_{\partial B_1} v(r\theta)(a_{ij}(r\theta)\theta_i\theta_j - 1) ds,$$

we can rewrite (35) as

$$(36) \quad \Delta v = \overline{\partial_i \partial_j (B_{ij}(v))} - \partial_i \partial_j (B_{ij}(v)) + \overline{\partial_i \partial_j (\phi_{ij}y)} - \partial_i \partial_j (\phi_{ij}y),$$

where

$$(37) \quad B_{ij}(v)(x) := (a_{ij}(x) - \delta_{ij}) \left(v(x) - \frac{1}{\alpha(r)} \oint_{\partial B_1} v(r\theta)(a_{ij}(r\theta)\theta_i\theta_j - 1) ds \right)$$

and

$$(38) \quad \phi_{ij}(x) = \frac{a_{ij}(x) - \delta_{ij}}{\alpha(r)}.$$

Using (3), it is clear that

$$(39) \quad M_p(B_{ij}(v), r) \leq c\Omega(r) M_p(v, r)$$

and

$$(40) \quad |\phi_{ij}(x)| \leq c\Omega(r).$$

These estimates indeed hold for $0 < r < \infty$, where we extend Ω to be zero for $r > 1$. Inverting the Laplacian by means of the fundamental solution, (36) becomes

$$(41) \quad v + Sv + Ty = 0,$$

where we may use Corollary 4 with (39) to obtain

$$(42) \quad M_p(Sv, r) \leq c \left(r \int_r^\infty \Omega(\rho) M_p(v, \rho) \rho^{-2} d\rho + r^{-n} \int_0^r \Omega(\rho) M_p(v, \rho) \rho^{n-1} d\rho \right)$$

and with (40) to obtain

$$(43) \quad M_p(Ty, r) \leq c \left(r \int_r^\infty \Omega(\rho) M_p(y, \rho) \rho^{-2} d\rho + r^{-n} \int_0^r \Omega(\rho) M_p(y, \rho) \rho^{n-1} d\rho \right).$$

To simplify the equations, let us introduce the function

$$(44) \quad E(r) = \exp \left(\int_r^\infty \frac{Q(t)}{t} dt \right),$$

where there is no problem with convergence of the integral since $Q(t) = 0$ for $t > 1$. Notice that $E(r)$ is continuous for $r \in (0, \infty)$, $E(r) \equiv 1$ for $r \geq 1$, and for any $r, \rho \in (0, \infty)$ we have

$$(45) \quad E^{-1}(r)E(\rho) = \exp \left(\int_\rho^r \frac{Q(t)}{t} dt \right).$$

To derive an estimate for this expression, let us use (5) with $\varepsilon \in (0, 1/2)$ and (9) to conclude

$$(46) \quad \delta \geq \int_{r/2}^r \frac{\Omega^2(t)}{t} dt \geq \Omega^2(r) r^{-2+2\varepsilon} \int_{r/2}^r t^{1-2\varepsilon} dt \geq c \Omega^2(r).$$

As a consequence of (33), we have

$$(47) \quad \left(\frac{\rho}{r}\right)^{c\sqrt{\delta}} \leq \left| \exp\left(\pm \int_{\rho}^r \frac{Q(t)}{t} dt\right) \right| \leq \left(\frac{r}{\rho}\right)^{c\sqrt{\delta}} \quad \text{for } 0 < \rho \leq r \leq 1.$$

Now let us express equations (28) and (41) in terms of the new dependent variables

$$(48) \quad Y(r) = E^{-1}(r)y(r) \quad \text{and} \quad V(x) = E^{-1}(|x|)v(x).$$

Since the operator K only involves integration in θ , it is clear that

$$(49) \quad Kv(r) = E(r)KV(r),$$

and so the equation (28) can be expressed as

$$(50) \quad Y'(r) = \frac{1}{r}KV(r).$$

We will be assuming below that $M_p(V, r) \leq c\Omega(r)$ for $0 < r < 1$, so $M_p(KV, r) \leq c\Omega^2(r)$, enabling us to integrate (50) to obtain

$$(51) \quad Y(r) = Y(0) + \int_0^r \frac{KV(\rho)}{\rho} d\rho.$$

The equation (41), on the other hand, is replaced by

$$(52) \quad V + S_1V + T_1Y = 0,$$

where

$$(53) \quad S_1 := E^{-1}SE \quad \text{and} \quad T_1 := E^{-1}TE,$$

with E representing the multiplication operator defined by the function (44).

Now let us substitute (51) into (52) to finally obtain the operator equation that we want to solve:

$$(54) \quad V + S_1V + T_2V = -Y(0)T_1(1),$$

where

$$(55) \quad T_2V(x) = E^{-1}(|x|)T_{\rho \rightarrow x} \left[E(\rho) \int_0^{\rho} KV(t) dt \right].$$

(In (54), notice that T_1 operates on functions of a single variable, say ρ , and $T_1(1)$ represents the action of T_1 on the function that is identically 1.) For a given choice of $Y(0)$, we can solve (54) uniquely for V provided we can show that the two integral operators involved have small norm on an appropriate function space. Consider the functions w in $L_{loc}^p(\mathbb{R} \setminus \{0\})$ for which the norm

$$(56) \quad \|w\|_{p,\Omega} := \sup_{0 < r < 1} \frac{M_p(w, r)}{\Omega(r)} + \sup_{r > 1} \frac{M_p(w, r)}{\sqrt{\delta} r^{-n}}$$

is finite, and take the closure to form a Banach space X . We want to show that the right hand side of (54) is in X and that the integral operators S_1 and T_2 map X to itself with small norm. It will be useful to observe that the continuity and positivity of $|E(r)|$ implies that, for any $w \in L_{loc}^p(\mathbb{R} \setminus \{0\})$, we have

$$(57) \quad M_p(Ew, r) = |E(\tilde{r})|M_p(w, r) \quad \text{for some } \tilde{r} = \tilde{r}_w \in (r, 2r),$$

with an analogous statement for E^{-1} .

To show $T_1(1) \in X$, we must estimate $M_p(T_1(1), r)$ separately for $0 < r < 1$ and $r > 1$. For $0 < r < 1$, we can use (57) to find $\tilde{r} \in (r, 2r)$ so that

$$M_p(T_1(1), r) = |E^{-1}(\tilde{r})| M_p(T_{\rho \rightarrow r}[E(\rho)], r),$$

and then use (43) to estimate

$$\begin{aligned} M_p(T_1(1), r) &\leq c |E^{-1}(\tilde{r})| \left(r \int_r^\infty \Omega(\rho) M_p(E, \rho) \rho^{-2} d\rho + r^{-n} \int_0^r \Omega(\rho) M_p(E, \rho) \rho^{n-1} d\rho \right) \\ &= c |E^{-1}(\tilde{r})| \left(r \int_r^\infty \Omega(\rho) |E(\tilde{\rho})| \rho^{-2} d\rho + r^{-n} \int_0^r \Omega(\rho) |E(\tilde{\rho})| \rho^{n-1} d\rho \right), \end{aligned}$$

where $\tilde{\rho} \in (\rho, 2\rho)$ by (57). But now we can use (45) and (47) to conclude

$$M_p(T_1(1), r) \leq c \left(r \int_r^1 \Omega(\rho) (\rho/r)^{c\sqrt{\delta}} \rho^{-2} d\rho + r^{-n} \int_0^r \Omega(\rho) (r/\rho)^{c\sqrt{\delta}} \rho^{n-1} d\rho \right)$$

Using the monotonicity properties (5) and (6), we obtain

$$M_p(T_1(1), r) \leq c \left(\Omega(r) r^{\varepsilon - c\sqrt{\delta}} \int_r^1 \rho^{-\varepsilon - 1 + c\sqrt{\delta}} d\rho + \Omega(r) r^{c\sqrt{\delta} - \varepsilon} \int_0^r \rho^{\varepsilon - c\sqrt{\delta} - 1} d\rho \right).$$

Provided δ is sufficiently small that $\varepsilon - c\sqrt{\delta} > 0$, we conclude that

$$(58) \quad M_p(T_1(1), r) \leq c \Omega(r) \quad \text{for } 0 < r < 1.$$

On the other hand, for $r > 1$ we use $E^{-1}(r) \equiv 1$ and $\Omega(r) \equiv 0$ to estimate

$$\begin{aligned} M_p(T_1(1), r) &= M_p(T_{\rho \rightarrow r}[E(\rho)], r) \leq c r^{-n} \int_0^1 \Omega(\rho) M_p(E, \rho) \rho^{n-1} d\rho \\ &= c r^{-n} \int_0^1 \Omega(\rho) |E(\tilde{\rho})| \rho^{n-1} d\rho \leq c r^{-n} \int_0^1 \Omega(\rho) \rho^{n - c\sqrt{\delta} - 1} d\rho. \end{aligned}$$

Provided $n - c\sqrt{\delta} > 0$, we obtain

$$(59) \quad M_p(T_1(1), r) \leq c \sqrt{\delta} r^{-n} \quad \text{for } r > 1.$$

The two estimates (58) and (59) together confirm that $T_1(1) \in X$.

Now let us show that S_1 maps X to itself with small operator norm. We suppose that $\|V\|_{p,\Omega} \leq 1$ and estimate $M_p(S_1 V, r)$ separately for $0 < r < 1$ and $r > 1$. For $0 < r < 1$ we have $M_p(V, r) \leq \Omega(r)$ and we can argue as in the previous paragraph to obtain

$$M_p(S_1 V, r) \leq c \left(r \int_r^1 \Omega^2(\rho) (\rho/r)^{c\sqrt{\delta}} \rho^{-2} d\rho + r^{-n} \int_0^r \Omega^2(\rho) (r/\rho)^{c\sqrt{\delta}} \rho^{n-1} d\rho \right).$$

Using (46) and the monotonicity analysis as above, we conclude that

$$(60) \quad \frac{M_p(S_1 V, r)}{\Omega(r)} \leq c\sqrt{\delta} \quad \text{for } 0 < r < 1.$$

For $r > 1$ we use (47) with $r = 1$ and (46) to obtain

$$M_p(S_1 V, r) \leq c r^{-n} \int_0^1 \Omega^2(\rho) \rho^{n - c\sqrt{\delta} - 1} d\rho \leq c r^{-n} \delta \int_0^1 \rho^{n - c\sqrt{\delta} - 1} d\rho \leq c r^{-n} \delta$$

provided δ is sufficiently small, and we conclude that

$$(61) \quad \frac{M_p(S_1 V, r)}{\sqrt{\delta} r^{-n}} \leq c\sqrt{\delta} \quad \text{for } r > 1.$$

The estimates (60) and (61) together show that S_1 maps X to itself with small operator norm.

Finally, we estimate T_2 . For $0 < r < 1$ and $M_p(V, r) \leq \Omega(r)$, we argue as before to write

$$\begin{aligned} M_p(T_2V, r) &\leq c \left(r \int_r^\infty \Omega(\rho) (\rho/r)^{c\sqrt{\delta}} M_p \left[\int_0^\rho KV(t) \frac{dt}{t}, \rho \right] \rho^{-2} d\rho \right. \\ &\quad \left. + r^{-n} \int_0^r \Omega(\rho) (r/\rho)^{c\sqrt{\delta}} M_p \left[\int_0^\rho KV(t) \frac{dt}{t}, \rho \right] \rho^{n-1} d\rho \right). \end{aligned}$$

But we can use (34) to obtain $M_p(KV, \rho) \leq c\Omega(\rho)M_p(V, \rho) \leq c\Omega^2(\rho)$ so that

$$\begin{aligned} M_p(T_2V, r) &\leq c \left(r \int_r^1 \Omega(\rho) (\rho/r)^{c\sqrt{\delta}} \left[\int_0^\rho \frac{\Omega^2(t)}{t} dt \right] \rho^{-2} d\rho + \right. \\ &\quad \left. r^{-n} \int_0^r \Omega(\rho) (r/\rho)^{c\sqrt{\delta}} \left[\int_0^\rho \frac{\Omega^2(t)}{t} dt \right] \rho^{n-1} d\rho \right) \leq c\delta\Omega(r). \end{aligned}$$

Using (46) and the monotonicity argument, we have

$$(62) \quad \frac{M_p(T_2V, r)}{\Omega(r)} \leq c\delta \quad \text{for } 0 < r < 1.$$

For $r > 1$, we use (47) with $r = 1$ and (33) to obtain

$$M_p(T_2V, r) \leq cr^{-n} \int_0^1 \Omega^2(\rho) \rho^{-c\sqrt{\delta}+n-2} d\rho \leq cr^{-n}\delta \int_0^1 \rho^{-c\sqrt{\delta}+n-2} d\rho \leq cr^{-n}\delta,$$

provided δ is sufficiently small. These estimates show that T_2 maps X to itself with small operator norm.

Since S_1 and T_2 have small operator norms on X , we conclude that (54) admits a unique solution $V \in X$. Let us now investigate the implications for the weak solution $Z(x) = \overline{Z}(|x|) + v(x)$ that we are trying to construct. Tracing back through the definitions, we see that our solution of (1) is given by

$$(63) \quad Z(x) = \frac{E(r)}{\alpha(r)} \left(Y(0) + Y_1(r) - \oint_{\partial B_1} V(r\theta) a_{ij}(r\theta) \theta_i \theta_j ds + \alpha(r)V(x) \right),$$

where

$$Y_1(r) = \int_0^r \frac{KV(\rho)}{\rho} d\rho.$$

Recall that $|\alpha(r) - 1| \leq c\Omega(r)$ and V satisfies $M_p(V, r) \leq c\Omega(r)$ as $r \rightarrow 0$. Moreover,

$$M_p \left[\int_0^r \frac{KV(\rho)}{\rho} d\rho, r \right] \leq c \int_0^r \frac{\Omega^2(\rho)}{\rho} d\rho,$$

so all the terms after the $Y(0)$ inside the parentheses of (63) are bounded in M_p either by $\Omega(r)$ or by $\int_0^r \Omega^2(\rho) \rho^{-1} d\rho$ for $0 < r < 1$. Let us explore $E(r)$. Notice that

$$\begin{aligned} \int_r^1 \frac{Q(\rho)}{\rho} d\rho &= \int_r^1 [-\alpha_0(\rho) + n\alpha(\rho)] \frac{d\rho}{\rho} + \int_r^1 Q(\rho)[1 - \alpha(\rho)] \frac{d\rho}{\rho} \\ &= \frac{1}{|\partial B_1|} \int_{B_1 \setminus B_r} (na_{ij}(y)y_i y_j |y|^{-2} - a_{ii}(y)) \frac{dy}{|y|^n} + \int_r^1 Q(\rho)[1 - \alpha(\rho)] \frac{d\rho}{\rho}. \end{aligned}$$

But $|Q(\rho)[1 - \alpha(\rho)]| \leq c\Omega^2(\rho)$, so

$$\int_r^1 Q(\rho)[1 - \alpha(\rho)] \frac{d\rho}{\rho} = C - \int_0^r Q(\rho)[1 - \alpha(\rho)] \frac{d\rho}{\rho}.$$

Exponentiating, we obtain

$$E(r) = \exp \left[\int_r^1 \frac{Q(\rho)}{\rho} d\rho \right] = C \exp \left[\frac{1}{|\partial B_1|} \int_{B_1 \setminus B_r} (na_{ij}(y)y_i y_j |y|^{-2} - a_{ii}(y)) \frac{dy}{|y|^n} \right] (1 + \zeta_0(r)),$$

where $|\zeta_0(r)| \leq c \int_0^r \Omega^2(\rho) \rho^{-1} d\rho$. Since we can rescale Z to set $CY(0) = 1$, we have the formula (10). This completes the proof of Theorem 1.

Remark 3. *Choosing $C = -|\partial B_1|^{-1}$ in the proof of Theorem 1 and modifying the rest of the argument, we could obtain the asymptotic representation of the fundamental solution of equation (1):*

$$\Gamma_{\mathcal{A}}(x) = \Gamma(x) + \Delta^{-1} \partial_i \partial_j ((a_{ij} - \delta_{ij}) \Gamma) + v(x),$$

where Γ is the fundamental solution of Δ and

$$M_p(v, r) \leq c \Omega^2(r) r^{2-n} \quad \text{for } r \in (0, 1).$$

Here the condition (4) is not necessary; we only need the smallness of $\Omega(r)$. However, this formula, as well as the additional terms in the asymptotic expansion of $\Gamma_{\mathcal{A}}$ can be obtained more easily by iteration from the equation $\mathcal{A}u(x) = \delta(x)$.

3. PROOF OF THEOREM 2

Let $q = p/(p-1)$ and $\beta, \gamma \in \mathbb{R}$. Let us introduce the weighted L^p norm for functions on \mathbb{R}^n with separate weights at the origin and infinity

$$(64) \quad \|u\|_{L_{\beta, \gamma}^q(\mathbb{R}^n)}^q = \|u\|_{L_{\beta}^q(B_1)}^q + \|u\|_{L_{\gamma}^q(B_1^c)}^q = \int_{|x| < 1} |u(x)|^q |x|^{\beta q} dx + \int_{|x| > 1} |u(x)|^q |x|^{\gamma q} dx,$$

and the weighted Sobolev space $W_{\beta, \gamma}^{2, q}(\mathbb{R}^n \setminus \{0\})$ with norm

$$(65) \quad \sum_{|\alpha| \leq 2} \|r^{|\alpha|} \partial^\alpha u\|_{L_{\beta, \gamma}^q(\mathbb{R}^n)}.$$

Notice that $L_{-\beta, -\gamma}^p(\mathbb{R}^n)$ is the dual space for $L_{\beta, \gamma}^q(\mathbb{R}^n)$ and the notation $W_{-\beta, -\gamma}^{-2, p}(\mathbb{R}^n \setminus \{0\})$ will be used for the dual of $W_{\beta, \gamma}^{2, q}(\mathbb{R}^n \setminus \{0\})$. Many authors have used similar weighted Sobolev spaces to study operators like the Laplacian on \mathbb{R}^n , $\mathbb{R}^n \setminus \{0\}$, and other noncompact manifolds with conical or cylindrical ends.

Using the analysis in [22], [21] or [20], for example, it is easily verified that the bounded operator

$$(66) \quad \Delta : W_{\beta, \gamma}^{2, q}(\mathbb{R}^n \setminus \{0\}) \rightarrow L_{\beta+2, \gamma+2}^q(\mathbb{R}^n)$$

is Fredholm (finite nullity and finite deficiency) for all values of β and γ *except* for the values $-2 + \frac{n}{p} + k$ and $-\frac{n}{q} - k$ where k is any nonnegative integer. In fact, (66) is an isomorphism for $-n/q < \beta, \gamma < -2 + n/p$ (recall that we are assuming $n \geq 3$, so such β, γ exist). Since we are principally interested in the behavior of functions at the origin, we will fix $\gamma_0 \in (-n/q, -2 + n/p)$. Then, for $\beta \in (-2 + n/p, -1 + n/p)$, we find that (66) is surjective with a one-dimensional nullspace spanned by $|x|^{2-n}$.

Now let us consider the formal adjoint of \mathcal{A} , which also defines a bounded operator on these spaces

$$(67) \quad \mathcal{L} = \bar{a}_{ij}(x) \partial_i \partial_j : W_{\beta, \gamma}^{2, q}(\mathbb{R}^n \setminus \{0\}) \rightarrow L_{\beta+2, \gamma+2}(\mathbb{R}^n),$$

where \bar{a}_{ij} , of course, denotes the complex conjugate of a_{ij} . Because $a_{ij}(x) = \delta_{ij}$ for $|x| > 1$ and $a_{ij}(x) - \delta_{ij}$ vanishes as $x \rightarrow 0$, the analysis in the above references shows that the operator (67) is Fredholm for exactly the same values of β and γ as for (66). In fact, for fixed nonexceptional values of β and γ , we may take δ sufficiently small and use perturbation theory (cf. [14], Ch.IV, Sec.5) to conclude that the nullity and deficiency of (66) and (67) agree.

So, in addition to the fixed $\gamma_0 \in (-n/q, -2 + n/p)$, let us now fix $\beta_1 \in (-n/q, -2 + n/p)$ and $\beta_2 \in (-2 + n/p, -1 + n/p)$, and denote the adjoints of the corresponding operators (67) by \mathcal{A}_1 and \mathcal{A}_2 :

$$(68) \quad \mathcal{A}_1 : L^p_{-\beta_1-2, -\gamma_0-2}(\mathbb{R}^n) \rightarrow W^{-2,p}_{-\beta_1, -\gamma_0}(\mathbb{R}^n \setminus \{0\})$$

is an isomorphism, and

$$(69) \quad \mathcal{A}_2 : L^p_{-\beta_2-2, -\gamma_0-2}(\mathbb{R}^n) \rightarrow W^{-2,p}_{-\beta_2, -\gamma_0}(\mathbb{R}^n \setminus \{0\})$$

is injective with a one-dimensional cokernel. An arbitrary non-zero functional in $\text{Coker } \mathcal{A}_2$ will be denoted by ζ .

We introduce a cut-off function $\eta \in C_0^\infty(B_1)$ equal to 1 on $B_{1/2}$. It follows from (63) that $\eta Z \in L^p_{-\beta_1-2}(B_1)$ but it is not in $L^p_{-\beta_2-2}(B_1)$. Let $F = \mathcal{A}((1 - \eta)Z) = -\mathcal{A}(\eta Z)$. Since $F = 0$ on $B_{1/2}$ and on B_1^c , it follows that $F \in W^{-2,p}_{-\beta_2, -\gamma_0}(\mathbb{R}^n \setminus \{0\}) \subset W^{-2,p}_{-\beta_1, -\gamma_0}(\mathbb{R}^n \setminus \{0\})$. But (68) is an isomorphism, so ηZ is the only solution of $\mathcal{A}_1 v = -F$. Since $L^p_{-\beta_2-2}(B_1) \subset L^p_{-\beta_1-2}(B_1)$, $\mathcal{A}_2 v = -F$ has no solutions, which means that $\zeta(F) \neq 0$.

Now let $u \in L^p_{\text{loc}}(B_1)$ be a weak solution of $\mathcal{A}u = 0$ satisfying $M_p(u, r) \leq c r^{2-n+\varepsilon_0}$. This estimate on u implies that $u \in L^p_{-\beta-2}(B_1)$ for $\beta < -n/q + \varepsilon_0$. So let us restrict our choice of β_1 to the interval $(-n/q, -n/q + \varepsilon_0)$. Denote by C an arbitrary constant. Since $\mathcal{A}(\eta(u - CZ)) = 0$ on $B_{1/2}$ and on B_1^c , we have $\mathcal{A}(\eta(u - CZ)) \in W^{-2,p}_{-\beta_2-2, -\gamma_0-2}(\mathbb{R}^n \setminus \{0\})$. Choosing C to satisfy

$$C\zeta(F) = \zeta(\mathcal{A}(\eta u)),$$

we obtain $\zeta(\mathcal{A}(\eta(u - CZ))) = 0$, which implies $\eta(u - CZ) \in \text{Dom}(\mathcal{A}_2)$, and in particular

$$\eta(u - CZ) \in L^p_{-\beta_2-2}(B_1).$$

But this implies that $w = u - CZ$ satisfies $M_p(w, r) \leq c r^{\beta_2+2-n/p}$. Assuming that we had fixed $\beta_2 = \frac{n}{p} - 1 - \varepsilon_1$ where $\varepsilon_1 \in (0, 1)$, we conclude that

$$(70) \quad M_p(w, r) \leq c r^{1-\varepsilon_1}.$$

This completes the proof.

REFERENCES

- [1] A.Aleksandrov, *Majorants of solutions of linear equations of order two*, Vestnik Leningrad University **21** 1966, 15-25.
- [2] I.Y.Bakelman, *Theory of quasilinear elliptic equations*, Siberian Math. J. **2** 1961, 179-186.
- [3] P.Bauman, *Positive solutions of elliptic equations in nondivergent form and their adjoints*, Ark. Mat. **22** 1984, 153-173.
- [4] P.Bauman, *Equivalence of the Green's function for diffusion operators in \mathbb{R}^n : a counterexample*, Proc. Amer. Math. Soc. **91** 1984, 64-68.
- [5] P.Bauman, *A Wiener test for nondivergence structure second-order elliptic equations*, Indiana U. Math. J. **34** 1985, 825-844.
- [6] L.Caffarelli, X.Cabr , *Fully Nonlinear Elliptic Equations*, AMS Colloquium Publications, Vol. 43, AMS, Providence, RI, 1995.
- [7] M.Cerutti, L.Escauriaza, E.Fabes, *Uniqueness in the Dirichlet problem for some elliptic operators with discontinuous coefficient*, Ann. Mat. Pura Appl. **CLXIII** 1993, 161-180.
- [8] L.Escauriaza, *Weak type-(1,1) inequalities and regularity properties of adjoint and normalized adjoint solutions to linear nondivergence form operators with VMO coefficients*, Duke Math. J. **74** 1994, 177-201.
- [9] L.Escauriaza, *Bounds for the fundamental solution of elliptic and parabolic equations in nondivergence form*, Comm. Part. Diff. Equat. **25** 2000, 821-845.
- [10] L.Escauriaza, C.Kenig, *Area integral estimates for solutions and normalized adjoint solutions to nondivergence form elliptic equations*, Ark. Mat. **31** 1993, 275-296.
- [11] E.Fabes, N.Garofalo, S.Marin-Malav , S.Salsa, *Fatou theorems for some nonlinear elliptic equations*, Rev. Mat. Ib. **4** 1988, 227-242.

- [12] E.Fabes, D.Stroock, *The L^p -integrability of Green's functions and fundamental solutions for elliptic and parabolic equations*, Duke Math. J. **51** 1984, 977-1016.
- [13] D.Gilbarg, N.Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.
- [14] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.
- [15] V.Kozlov, V.Maz'ya, *The estimates for L_p -averages and asymptotics of solutions of elliptic boundary value problems in a cone I* (Russian), Seminar Analysis Operator Equat. and Numer. Anal. 1985/86 Karl-Weierstrass-Institut für Mathematik, 55-92,
- [16] V.Kozlov, V.Maz'ya, *The estimates for L_p -averages and asymptotics of solutions of elliptic boundary value problems in a cone II* (Russian), Math. Nachr. **137** 1980, 113-139.
- [17] V.Kozlov, V.Maz'ya, *Differential Equations with Operator Coefficients*, Springer Monographs in Mathematics, Springer-Verlag, 1999.
- [18] V.Kozlov, V.Maz'ya, *Asymptotic formula for solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients near the boundary* (English summary), Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** 2003, 551-600.
- [19] N.Krylov, M.Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Math. USSR Izv. **16** 1981, 151-164.
- [20] R.Lockhart, R.McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **12** 1985, 409-447.
- [21] R.McOwen, *The behavior of the Laplacian on weighted Sobolev spaces*, Comm. Pure Appl. Math. **32** 1979, 783-795.
- [22] V.Maz'ya, B.Plamenevski, *Estimates in L_p and Hölder classes and the Miranda-Agmon maximum principle solutions of elliptic boundary problems in domains with singular points on the boundary* (in Russian), Math. Nachr. **81** 1978, 25-82.
- [23] C.Pucci, *Limitazioni per soluzioni di equazioni ellittiche*, Ann. Mat. Pura Appl. **74** 1966, 15-30.
- [24] P.Sjögren, *On the adjoint of an elliptic linear differential operator and its potential theory*, Ark. Mat. **11** 1973, 153-165.
- [25] D.Strook, S.Varadhan, *Diffusion processes with continuous coefficients, I*, Comm. Pure Appl. Math. **22** 1969, 345-400.
- [26] D.Strook, S.Varadhan, *Diffusion processes with continuous coefficients, II*, Comm. Pure Appl. Math. **22** 1969, 479-530.
- [27] E.Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

LINKÖPING UNIVERSITY, OHIO STATE UNIVERSITY, UNIVERSITY OF LIVERPOOL

NORTHEASTERN UNIVERSITY